SOME PROBLEMS OF CREDIT MIGRATION IN THE CONTEXT OF SOLVENCY II

Tomas CIPRA
Charles University of Prague
cipra@karlin.mff.cuni.cz

1. INTRODUCTION

- debtors of banks and insurance companies are typically classified by means of various ***credit ratings***

- the credit evaluating methods based on (external or internal) ratings should respect the possibility of ***credit migration*** in time

Credit ratings present a typical approach to quantification of credit risk in ***Solvency II*** and ***Basel II***:

- ***Standardized approach***:
  - methods based on external ratings
  - risk weights are assigned according to external rating
  - external agencies: Standard & Poor’s, Moody’s, Fitch, ...
- ***Internal ratings based approach***:
  - methods based on external ratings
  - capital demand is evaluated by specific formulas
  - two forms:
    - ***foundation***: the institution evaluates only the default probability
    - ***advanced***: the institution evaluates also other para-meters (expected loss, ...)
  - boths forms are elaborated by Czech banks and insurance companies nowadays

The credit evaluating methods based on (external or internal) ratings should respect the possibility of ***credit migration*** in time:

- credit migration is typical for Czech Republic due to high dynamics of loans commenced in the previous decade with strong acceleration nowadays (due to dynamic development of economics in Czech Republic)

Suitable instruments for the description of rating process in time which enable to respect credit migration in a realistic way are ***Markov chains***

2. BASIC FACTS ON MIGRATION MATRICES

Markov chain model for credit migration:

– finite state space, e.g. (S&P):

$$\{AAA, AA, A, BBB, BB, B, CCC, D(efault)\}$$

– ***migration*** (or ***transition***) ***probabilities***:

$$m_{ij} = \mathrm{P}(i \to j), \quad i, j = 1, \ldots, 8$$

= probability of change from rating class i at the beginning of a year to rating class j at year's end

Properties of matrix $M = (m_{ij})_{i,j=1,\dots,8}$:

- $m_{ij} \geq 0, \quad i, j = 1, \dots, 8$
- default state D is absorbing: $m_{8,j} = 0,\ j = 1, \dots, 7; \quad m_{8,8} = 1$
- last column contains 1-year probabilities of default PD
- $\sum_{j=1}^{8} m_{ij} = 1, \quad i = 1, \dots, 8$ (***stochastic matrix***)
- $m_{i,8} \leq m_{i+1,8}, \quad i = 1, \dots, 7$

  (low-risk states never show a higher PD than high-risk states)
- $\dots \leq m_{i,i-2} \leq m_{i,i-1} \leq m_{ii} \geq m_{i,i+1} \geq m_{i,i+2} \geq \dots$

  (***row monotony towards the diagonal***)
- $\dots \leq m_{i-2,i} \leq m_{i-1,i} \leq m_{ii} \geq m_{i+1,i} \geq m_{i+2,i} \geq \dots$

  (***column monotony towards the diagonal***)
- $\sum_{j \geq k} m_{ij}$ is a nondecreasing function of i for every fixed k

  (***stochastic monotony***)

Generator matrix:

– the instrument to embed the time-discrete Markov chain in a time-continuous Markov process (e.g. to control under-year interventions)

– ***generator matrix*** Q enables to control time-dependent migration matrices $M(t)$ by means of differential equation:

$$\mathrm{d}M(t) = Q \cdot M(t)\,\mathrm{d}t$$

under the boundary condition $M(0) = \mathrm{I}$

– solution: $$M(t) = \mathrm{e}^{t \cdot Q} = \sum_{k=0}^{\infty} \frac{(tQ)^k}{k!}$$

– properties of generator matrix $Q = (q_{ij})_{i,\,j=1,\,\dots,\,8}$:

- $q_{ii} \leq 0, \quad i = 1, \dots, 8$
- $q_{ij} \geq 0, \quad i \neq j$
- $\sum_{j=1}^{8} q_{ij} = 0, \quad i = 1, \dots, 8$

The problem of finding a generator for an empirical migration matrix M:

– ***theory***:

if $m_{ii} > 1/2$ for all i (i.e. M is ***strictly diagonally dominant***) then

$$\widetilde{Q} = \sum_{k=0}^{\infty} (-1)^{k+1} \frac{(M - \mathrm{I})^k}{k}$$

is geometrically convergent with $\widetilde{Q}$ having row-sums zero and $\exp(\widetilde{Q}) = M$ (i.e. $\widetilde{Q}$ is the matrix logarithm of M)

– ***practice*** (off-diagonal entries $\tilde{q}_{ij}$ should be nonnegative):

– approximation: $\tilde{q}_{ii} = \ln(m_{ii}), \quad \tilde{q}_{ij} = \frac{m_{ij}\ln(m_{ii})}{m_{ii}-1}, \ i \neq j$

– approximation (using $\tilde{Q} = \ln(M)$):

$$\tilde{\tilde{q}}_{ii} = \tilde{q}_{ii} + \sum_{i \neq j} \min(\tilde{q}_{ij}, 0), \quad \tilde{\tilde{q}}_{ij} = \max(\tilde{q}_{ij}, 0), \ i \neq j$$

– adjustment of the generator to reproduce a given default column (i.e. to reproduce given *Pd*s) is possible

then:

$$M(t) = \mathrm{I} + t\tilde{Q} + \frac{(t\tilde{Q})^2}{2!} + \ldots, \quad \text{resp. } M(t) = \mathrm{I} + t\tilde{\tilde{Q}} + \frac{(t\tilde{\tilde{Q}})^2}{2!} + \ldots$$

Migration probabilities*: *estimation and testing

Markov chain models of credit migration may not be ***homogenous***:

- e.g. ***aging phenomenon***: migration probabilities for particular rating classes different from the initial one increase with aging of bonds

Simple ***cohort estimation*** for homogenous Markov chain (it can be interpreted as ML estimation):

$$\hat{m}_{ij} = \frac{n_{ij}}{\sum_k n_{ik}} = \frac{n_{ij}}{n_i}$$

n_{ij} number of migrations from rating i to rating j within one year

n_i number of migrations from rating i within one year

Cumulative cohort estimation over T-year period:

$$\hat{m}_{ij}(T) = \frac{n_{ij}(T)}{\sum_k n_{ik}(T)} = \frac{n_{ij}(T)}{n_i(T)}$$

$n_{ij}(T)$ cumulative number of one-year migrations from rating i to rating j within T years

Test of homogenity (i.e. ***test of time invariance***) for migration probabilities:

H_0: $m_{ij}(0, 1) = m_{ij}(1, 2) = m_{ij}(T-1, T)$ for all i, j

$m_{ij}(t, t+1)$	probability of migration from rating i to rating j within the year $(t, t+1)$ with cohort estimation $\hat{m}_{ij}(t,t+1)$
$n_i(t, t+1)$	number of one-year migrations from rating i within the year $(t, t+1)$

– test statistics:

$$X_i^2 = \sum_{j=1}^{8} \sum_{t=0}^{T-1} \frac{\left[\hat{m}_{ij}(t,t+1) - n_i(t,t+1)\cdot \hat{m}_{ij}(T)\right]^2}{n_i(t,t+1)\cdot \hat{m}_{ij}(T)} \sim \chi^2_{7\cdot(T-1)} \text{ asympt.}$$

– if the null hypothesis H_0 cannot be rejected the migration is homogenous and one can use estimates $\hat{m}_{ij} = \hat{m}_{ij}(T)$ for all i, j

Non-Markovian behavior can be estimated using e.g. approach by Jafry and Schuermann (2003)

3. CreditMetrics APPROACH

Credit change indicator:

– CreditMetrics approach assumes that rating transitions reflect an underlying continuous ***credit-change indicator*** X

– it enables to simplify the description of credit migration process by means of “probability bins“

Let X have the standard normal distribution N(0, 1) with distribution function Φ: then conditional on an initial credit i one partitions X values into a set of disjoint bins $(x_j^i, x_{j+1}^i]$ so that the probability that X falls within a given interval equals the corresponding migration probability m_{ij} (estimated usually from historical data):

$$P(X \in (x_j^i, x_{j+1}^i]) = \Phi(x_{j+1}^i) - \Phi(x_j^i) = m_{ij}$$

(e.g.: the lowest default bin has a lower threshold $-\infty$,
the highest *AAA* bin has an upper threshold $+\infty$)

4. CREDIT CYCLES

– analysis of credit cycles may substantially improve the accurate assessment of credit risk

– the simplest way to measure the credit cycle is by means of a ***one factor*** Z_t meaning “the values of default rates and of end-of-period risk ratings not predicted (using historical average transition rates) by the initial mix of credit grades“

– the credit-change indicator X can be decomposed into two parts:

$$X = \sqrt{1-\rho} \cdot Y + \sqrt{\rho} \cdot Z$$

Y ***idiosyncratic component*** unique to a borrower ($Y \sim N(0, 1)$)

Z ***systematic component*** shared by all borrowers ($Z \sim N(0, 1)$)

ρ positive parameter representing correlation between Y and Z (Y and Z are mutually independent so that Z explains a fraction ρ of the variance of X)

– for a fixed ρ and t one can estimate Z_t using least squares principle:

$$\min_{Z_t} \sum_i \sum_j \frac{n_i(t,t+1)\left(m_{ij}(t,t+1) - \Delta(x^i_{j+1}, x^i_j, Z_t)\right)^2}{\Delta(x^i_{j+1}, x^i_j, Z_t) \cdot \left(1 - \Delta(x^i_{j+1}, x^i_j, Z_t)\right)}$$

where

$$\Delta(x^i_{j+1}, x^i_j, Z_t) = \Phi\left(\frac{x^i_{j+1} - \sqrt{\rho} \cdot Z_t}{\sqrt{1-\rho}}\right) - \Phi\left(\frac{x^i_j - \sqrt{\rho} \cdot Z_t}{\sqrt{1-\rho}}\right)$$

– one can find the appropriate value of ρ such that the series $\{Z_t\}$ has the unit variance

Interpretation (for a given initial credit rating):

– $Z_t > 0$ in good years:
  a lower than average *PD*
  a higher than average ratio of upgrades to downgrades

– $Z_t < 0$ in bad years:
  a higher than average *PD*
  a lower than average ratio of upgrades to downgrades

– one can generalize such an analysis to ***multi-factor*** credit migrations (see e.g. Wei (2000))

5. RISK-NEUTRAL PROBABILITY VALUATON MODEL

– ***martingale approach*** to pricing securities can be used also in this context

– it derives a ***risk premium*** for the dynamic credit rating process (i.e. for the corresponding Markov chain)

– the transition matrix $M = (m_{ij})_{i,j=1,\ldots,8}$ becomes under a ***risk-neutral probability measure*** the form $P(t, t+1)$:

$$P(t,t+1) = \left(p_{ij}(t,t+1)\right) = \left(\pi_{ij}(t) \cdot m_{ij}\right)$$

To estimate the risk premiums $\pi_{ij}(t)$ one can use bond price data and recovery rates:

$v_0(t,T)$	time-t price of a riskless unit discount bond maturing at time T
$v_i(t,T)$	time-t price of a unit discount bond at the rating class i maturing at time T
δ	recovery fraction of the par in the case of default

Possible choice of risk premiums $\pi_{ij}(t)$:

– $p_{ij}(t,t+1) = \pi_i(t)\cdot m_{ij},\ i \neq j, \quad p_{ii}(t,t+1) = 1 - \sum_{j\neq i} \pi_i(t)\cdot m_{ij}$

– then $\pi_i(0) = \dfrac{v_0(0,1) - v_i(0,1)}{(1-\delta)\cdot v_0(0,1)\cdot m_{i8}}$

– it enables recursive calculation of risk neutral probabilities $P(t, t+1)$ and $P(0, t)$ (it holds $P(0, t) = P(0, 1)\cdot \ldots \cdot P(t-1, t)$)

Another possible choice of risk premium $\pi_{ij}(t)$ when $m_{i8} \approx 0$:

– $p_{ij}(t,t+1) = \pi_i(t) \cdot m_{ij}, \;\; j \neq 8, \quad p_{i8}(t,t+1) = 1 - \pi_i(t)(1 - m_{i8})$

– then $\pi_i(0) = \dfrac{v_i(0,1) - \delta \cdot v_0(0,1)}{(1-\delta) \cdot v_0(0,1) \cdot (1 - m_{i8})}$

– it enables recursive calculation of risk neutral probabilities $P(t, t+1)$ and $P(0, t)$ as in the previous case

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