SOME PROBLEMS OF CREDIT MIGRATION IN THE CONTEXT OF SOLVENCY II

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This contribution:
- *summarizes some known basic facts on Markov chains in the context of credit migration;*
- *selects among various methods which enable to describe credit migration the ones that may be suitable from the point of view of Solvency II for insurance institutions.*

1. Introduction

Credit ratings by means of which are classified debtors should be important instruments of internal models in the framework of Solvency II and Basel II (see e.g. Sandström (2005), Schubert and Griessmann (2005)).

Such ratings in practice of banks and insurance institutions are connected with many problems both of practical and theoretical character. This paper concentrates on a special problem in this context due to credit migration in time since the credit evaluating methods based on (external or internal) ratings should respect the possibility of changes in time.

In general, quantification of credit risk can be performed by
- *standardized approach* that includes methods based on external ratings by means of risk weights that are assigned by external agencies (Standard & Poor's, Moody's, Fitch, Best and others; some of them have branch offices in Czech Republic now);
- *internal ratings based approach* that includes capital demand evaluation by specific formulas.

Suitable instruments for the description of rating process in time which enable to respect credit migration in a realistic way are Markov chains.

2. Basic facts on migration matrices

Let us consider a simple *Markov chain model for credit migration* with a finite state space, e.g. according to S&P:

$$\{AAA, AA, A, BBB, BB, B, CCC, D(efault)\}$$

Then we work with *migration* (or *transition*) *probabilities*:

$$m_{ij} = \mathrm{P}(i \to j), \quad i, j = 1, ..., 8$$

which are probabilities of change from the rating class i at the beginning of a year to the rating class j at the year's end.

The matrix $M = (m_{ij})_{i,j=1, ..., 8}$ has the following properties:

- $m_{ij} \geq 0, \quad i, j = 1, ..., 8$;
- default state D is absorbing: $m_{8,j} = 0, \ j = 1, ..., 7; \quad m_{8,8} = 1$;
- last column contains 1-year probabilities of default PD;
- $\sum_{j=1}^{8} m_{ij} = 1, \quad i = 1, ..., 8$ (*stochastic matrix*);
- $m_{i,8} \leq m_{i+1,8}, \quad i = 1, ..., 7$
  (low-risk states never show a higher PD than high-risk states);
- $... \leq m_{i,i-2} \leq m_{i,i-1} \leq m_{ii} \geq m_{i,i+1} \geq m_{i,i+2} \geq ...$
  (*row monotony towards the diagonal*);
- $... \leq m_{i-2,i} \leq m_{i-1,i} \leq m_{ii} \geq m_{i+1,i} \geq m_{i+2,i} \geq ...$
  (*column monotony towards the diagonal*);
- $\sum_{j \geq k} m_{ij}$ is a nondecreasing function of i for every fixed k
  (*stochastic monotony*).

Generator matrix Q is the instrument to embed the time-discrete Markov chain in a time-continuous Markov process (e.g. to control under-year interventions). It enables to control time-dependent migration matrices $M(t)$ by means of differential equation:

$$\mathrm{d}M(t) = Q \cdot M(t) \ \mathrm{d}t$$

under the boundary condition $M(0) = \mathrm{I}$. The equation has a solution

$$M(t) = \mathrm{e}^{t \cdot Q} = \sum_{k=0}^{\infty} \frac{(tQ)^k}{k!} \quad .$$

The generator matrix $Q = (q_{ij})_{i,j=1, ..., 8}$ has the following properties:

- $q_{ii} \leq 0, \quad i = 1, ..., 8$;
- $q_{ij} \geq 0, \quad i \neq j$;
- $\sum_{j=1}^{8} q_{ij} = 0, \quad i = 1, ..., 8$.

The problem of finding a generator for an empirical migration matrix M can be solved theoretically: if $m_{ii} > 1/2$ for all i (i.e. M is *strictly diagonally dominant*) then

$$\widetilde{Q} = \sum_{k=0}^{\infty} (-1)^{k+1} \frac{(M - \mathrm{I})^k}{k}$$

is geometrically convergent with $\widetilde{Q}$ having row-sums zero and $\exp(\widetilde{Q}) = M$ (i.e. $\widetilde{Q}$ is the matrix logarithm of M).

In practice, off-diagonal entries $\widetilde{q}_{ij}$ should be nonnegative, and therefore one uses approximations:

- $\widetilde{q}_{ii} = \ln(m_{ii}), \quad \widetilde{q}_{ij} = \dfrac{m_{ij} \ln(m_{ii})}{m_{ii} - 1}, \; i \neq j$;
- $\widetilde{\widetilde{q}}_{ii} = \widetilde{q}_{ii} + \sum_{i \neq j} \min(\widetilde{q}_{ij}, 0), \quad \widetilde{\widetilde{q}}_{ij} = \max(\widetilde{q}_{ij}, 0), \; i \neq j$, where $\widetilde{Q} = \ln(M)$

(moreover, an adjustment of the generator to reproduce a given default column, i.e. to reproduce given probabilities of default, is possible). Then

$$M(t) = \mathrm{I} + t\widetilde{Q} + \frac{(t\widetilde{Q})^2}{2!} + \ldots, \quad \text{resp. } M(t) = \mathrm{I} + t\widetilde{\widetilde{Q}} + \frac{(t\widetilde{\widetilde{Q}})^2}{2!} + \ldots$$

3. Migration probabilities: estimation and testing

Markov chain models of credit migration may not be *homogenous* (e.g. there can be an *aging phenomenon*, i.e.migration probabilities for particular rating classes different from the initial one increase with aging of bonds).

Simple *cohort estimation* for homogenous Markov chain that can be interpreted as ML estimation has the form

$$\hat{m}_{ij} = \frac{n_{ij}}{\sum_k n_{ik}} = \frac{n_{ij}}{n_i},$$

where n_{ij} is the number of migrations from rating i to rating j within one year and n_i is number of migrations from rating i within one year.

Cumulative cohort estimation over T-year period is then

$$\hat{m}_{ij}(T) = \frac{n_{ij}(T)}{\sum_k n_{ik}(T)} = \frac{n_{ij}(T)}{n_i(T)},$$

where $n_{ij}(T)$ is the cumulative number of one-year migrations from rating i to rating j within T years and similarly for $n_i(T)$.

Test of homogenity (i.e. *test of time invariance*) for migration probabilities is the test of null hypothesis

$$H_0: \quad m_{ij}(0,1) = m_{ij}(1,2) = m_{ij}(T-1,T) \quad \text{for all } i, j,$$

where $m_{ij}(t, t+1)$ is the probability of migration from rating i to rating j within the year $(t, t+1)$ with cohort estimation $\hat{m}_{ij}(t,t+1)$ and $n_i(t, t+1)$ is the number of one-year migrations from rating i within the year $(t, t+1)$. Then the corresponding test statistic

$$X_i^2 = \sum_{j=1}^{8} \sum_{t=0}^{T-1} \frac{\left[\hat{m}_{ij}(t,t+1) - n_i(t,t+1)\cdot \hat{m}_{ij}(T)\right]^2}{n_i(t,t+1)\cdot \hat{m}_{ij}(T)} \sim \chi^2_{7\cdot(T-1)} \text{ asympt.}$$

under null hypothesis. If the null hypothesis H_0 cannot be rejected then the migration is homogenous and one can use estimates $\hat{m}_{ij} = \hat{m}_{ij}(T)$ for all i, j. *Non-Markovian behavior* can be estimated using e.g. approach by Jafry and Schuermann (2003).

4. CreditMetrics approach

CreditMetrics approach assumes that rating transitions reflect an underlying continuous *credit-change indicator* X. It enables to simplify the description of credit migration process by means of "probability bins".

Let X have the standard normal distribution N(0, 1) with distribution function Φ. Then conditionally on an initial credit i one partitions X values into a set of disjoint bins $(x_j^i, x_{j+1}^i]$ so that the probability that X falls within a given interval equals the corresponding migration probability $\hat{m}_{ij}$ (estimated usually from historical data):

$$P(X \in (x_j^i, x_{j+1}^i]) = \Phi(x_{j+1}^i) - \Phi(x_j^i) = \hat{m}_{ij}$$

(e.g., the lowest default bin has a lower threshold $-\infty$, the highest *AAA* bin has an upper threshold $+\infty$).

5. Credit cycles

Credit cycles may substantially improve the accurate assessment of credit risk. The simplest way to measure the credit cycle is by means of a *one factor* Z_t meaning "the values of default rates and of end-of-period risk ratings not predicted (using historical average transition rates) by the initial mix of credit grades".

The credit-change indicator X can be decomposed into two parts:

$$X = \sqrt{1-\rho}\cdot Y + \sqrt{\rho}\cdot Z$$

where Y is an *idiosyncratic component* unique to a borrower ($Y \sim$ N(0, 1)), Z is a *systematic component* shared by all borrowers ($Z \sim$ N(0, 1)) and ρ is a positive parameter representing correlation between X and Z (Y and Z are mutually independent so that Z explains a fraction ρ of the variance of X).

For a fixed ρ and t one can estimate Z_t using least squares principle:

$$\min_{Z_t} \sum_i \sum_j \frac{n_i(t,t+1)\left(m_{ij}(t,t+1) - \Delta(x^i_{j+1}, x^i_j, Z_t)\right)^2}{\Delta(x^i_{j+1}, x^i_j, Z_t)\cdot\left(1-\Delta(x^i_{j+1}, x^i_j, Z_t)\right)},$$

where

$$\Delta(x^i_{j+1}, x^i_j, Z_t) = \Phi\left(\frac{x^i_{j+1} - \sqrt{\rho}\cdot Z_t}{\sqrt{1-\rho}}\right) - \Phi\left(\frac{x^i_j - \sqrt{\rho}\cdot Z_t}{\sqrt{1-\rho}}\right)$$

(one can find the appropriate value of ρ such that the series $\{Z_t\}$ has the unit variance).

Interpretation (for a given initial credit rating) can be following:

- $Z_t > 0$ in good years:
  it means a lower than average probability of default and a higher than average ratio of upgrades to downgrades;
- $Z_t < 0$ in bad years:
  it means a higher than average probability of default and a lower than average ratio of upgrades to downgrades.

One can generalize such an analysis to *multi-factor* credit migrations (see e.g. Wei (2000)).

6. Risk-neutral probability valuation model

Martingale approach to pricing securities can be used also in this context: it derives *risk premiums* for the dynamic credit rating process (i.e. for the corresponding Markov chain).

The transition matrix $M = (m_{ij})_{i,j=1,\dots,8}$ retains under a *risk-neutral probability measure* the form $P(t, t+1)$:

$$P(t,t+1) = \left(p_{ij}(t,t+1)\right) = \left(\pi_{ij}(t)\cdot m_{ij}\right).$$

To estimate the risk premiums $\pi_{ij}(t)$ one can use bond price data and recovery rates:

$v_0(t,T)$ time-t price of a riskless unit discount bond maturing at time T;
$v_i(t,T)$ time-t price of a unit discount bond at the rating class i maturing at time T;
δ recovery fraction of the par in the case of default.

Possible choice of risk premiums $\pi_{ij}(t)$:

- $p_{ij}(t,t+1) = \pi_i(t)\cdot m_{ij},\ i \neq j, \quad p_{ii}(t,t+1) = 1 - \sum_{j\neq i} \pi_i(t)\cdot m_{ij}$;
- then $\pi_i(0) = \dfrac{v_0(0,1) - v_i(0,1)}{(1-\delta)\cdot v_0(0,1)\cdot m_{i8}}$;

– it enables recursive calculation of risk neutral probabilities $P(t, t+1)$ and $P(0, t)$ (it holds $P(0, t) = P(0, 1) \cdot ... \cdot P(t-1, t)$).

Another possible choice of risk premium $\pi_{ij}(t)$ when $m_{i8} \approx 0$:

– $p_{ij}(t,t+1) = \pi_i(t) \cdot m_{ij},\ \ j \neq 8, \quad p_{i8}(t,t+1) = 1 - \pi_i(t)(1 - m_{i8})$;
– then $\pi_i(0) = \dfrac{v_i(0,1) - \delta \cdot v_0(0,1)}{(1-\delta) \cdot v_0(0,1) \cdot (1 - m_{i8})}$;
– it enables recursive calculation of risk neutral probabilities $P(t, t+1)$ and $P(0, t)$ as in the previous case.

The work is a part of research project MSM0021620839 financed by Ministry of Education of Czech Republic

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